Identification of Random Variation in Structures and Their Parameter Estimates

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Abstract

Structures that are members of an ensemble of nominally identical systems actually differ due to variations in details among individuals. Furthermore, there are variations in the system response of an individual structure that can be attributed to unmeasured conditions (such as temperature and humidity) that are present during experiments. Finally, noise is present in all measurements of structural excitations and responses. For these reasons, there is always random variation associated with the characterizations of structural dynamic systems, and descriptions of results must be in statistical or probabilistic terms. This study identifies and assesses the sources and the degrees of randomness in a metric of structural dynamics of a given system through experiments and analysis.

Nomenclature

h = an n x 1 vector random source of Impulse Response Function realizations

$\mathbf{h}_i$ = an IRF realization of vector random source **h**

q = structural excitation

v = an n x 1 real valued vector

$\bar{\mathbf{v}}$ = sample mean of a collection of vectors in **v**

$\|\mathbf{v}\|$ = magnitude of vector **v**

x = structural response

H = structural frequency response function

P = normalized inner product between vectors $\mathbf{v}_1$ and $\mathbf{v}_2$

V = n x 1 vector of random variables

δ = distance between normalized vectors $\mathbf{v}_1$ and $\mathbf{v}_2$

$\overline{\delta^2}$ = mean square estimator of a random variable

ϕ = angle between vectors $\mathbf{v}_1$ and $\mathbf{v}_2$

θ = Fourier transform of the structural excitation

$\hat{\sigma}_\delta$ = root mean square (RMS) estimator of a random variable

$\hat{\sigma}_\Delta$ = RMS distance of vectors in **V** from their mean vector

ξ = Fourier transform of the structural response

Δ = random variable of which δ_i, i= 1,…,M, are realizations

1. Introduction

Practically all physical structures are subjected to dynamic environments. The dynamic response of some structures governs their design and therefore, characterizing the dynamic behavior of structures is important. However, the dynamic characteristics of physical structures that are nominally identical exhibit random variation (see Paez, Hunter, and Cafeo, 2002 [5], [6]). Specifically, manufacturing processes that seek to produce identical systems fail to do so because the equipment used to fabricate parts and the procedures used to construct components and systems are imperfect. The small variations in hardware and fabrication procedures yield systems with differing vibration characteristics.

Even if physical systems could be constructed identically, their behaviors would display random variation due to the influence of randomly varying boundary and initial conditions. Material temperature also influences structural behavior and is not normally taken into account in dynamic analysis or the interpretation of measured response data. Finally, the transducers used to measure system behavior are always noisy. Therefore, the excitations and responses used to infer system characteristics are only approximations. All these sources of variability result in structural randomness.

This paper attempts to quantify variation in a metric of structural response associated with nominally identical structures. Traditional vibration analysis techniques are used to estimate structural impulse response functions (IRF), which will be used to assess randomness in structural response. The methods of statistics are used to quantify structural variations by estimating a metric of differences between IRFs. Several different sources of variability associated with the structural response of a system are examined. A simple frame structure is used to demonstrate the techniques developed in this study. A combined experimental and numerical example is presented.

2. Experimental Structural Dynamics

Every physical structure subjected to dynamic loads realizes a response that depends on the magnitude and type of input to the structure as well as the characteristics of the structure. In this study, we sought to assess the variation of experimental structures in terms of their impulse response functions (IRF). Therefore, we required a means for estimating this function given the excitation and response.

A linear framework was assumed for modeling the behavior of the system of interest. Further, the system characteristics were assumed to be stable over the short time when a sequence of excitations was applied in immediate succession. Finally, the measured excitation and response of the experimental system were assumed to be available. The $(i)^{th}$ measurement of a scalar excitation is denoted as $q_i(t)$, $i=1,\ldots,M$, $t \geq 0$. The response excited by this input is denoted as $x_i(t)$, $i=1,\ldots,M$, $t \geq 0$. The Fourier Transforms of these signals are denoted as, $\theta_i(f)$, $-\infty < f < \infty$, and $\xi_i(f)$, $-\infty < f < \infty$, respectively. The relation between excitation and response is defined as

$$\xi_i(f) = H(f)\theta_i(f)\,, -\infty < f < \infty\,, i = 1,\ldots,M \qquad (1)$$

where H(f) is the system frequency response function (FRF).

To estimate the FRF, a standard procedure was used (see for example Wirshing, Paez, and Ortiz, 1995 [7] , or Bendat and Piersol, 1986 [2]), where both sides of Eq. (1) are multiplied by the complex conjugate of $\theta_i(f)$, and then the results are summed and the FRF is factored out:

$$\sum_{i=1}^{M} \xi_i(f)\theta_i^*(f) = H(f)\sum_{i=1}^{M} \left|\theta_i(f)\right|^2\,,\ -\infty < f < \infty \qquad (2)$$

Then we solved for the FRF as follows:

$$H(f) = \frac{\sum_{i=1}^{M} \xi_i(f)\theta_i^*(f)}{\sum_{i=1}^{M} \left|\theta_i(f)\right|^2}\,,\ -\infty < f < \infty \qquad (3)$$

The estimation formula used here minimizes the effects of transducer noise.

In practice, we do not have measurements of the excitation and response in continuous time. Rather, both excitation and response are sampled and denoted, respectively, as $q_{j,i}$, $x_{j,i}$, $j=0,\ldots,n-1$, $i=1,\ldots M$, where measurements were made at time $t_j=j\Delta t$, $j=0,\ldots,n-1$. A discrete frequency approximation of the FRF was established by evaluating Eq (3) at discrete frequencies $f_k=k\Delta f$, $k=0,\ldots,n/2$, $\Delta f=1/(n\Delta t)$.

$$H_k = H(f_k) = \frac{\sum_{i=1}^{M} \xi_i(f_k)\theta_i^*(f_k)}{\sum_{i=1}^{M} \left|\theta_i(f_k)\right|^2}\ ,\ k = 0,\ldots,\frac{n}{2} \qquad (4)$$

This expression for the FRF yields the fundamental measure of linear system behavior.

The IRF can then be obtained from the FRF via inverse Discrete Fourier Transform:

$$h_j = \frac{1}{\Delta t}\left[\frac{1}{n}\sum_{k=0}^{n-1} H_k e^{i2\pi jk/n}\right],\ j = 0,\ldots,n-1 \qquad (5)$$

where the values of H_k for $k=n/2,\ldots,n-1$, are the conjugate symmetric pairs of the values in $k=1,\ldots,n/2$.

3. Statistical Analysis

The objective in this analysis was to assess the degree of system random variation that results from structural differences. This goal was accomplished using the system impulse response function (IRF) as the critical measure of structural behavior. Simple statistical methods such as the mean and root mean square (RMS) were employed to evaluate system random variation. The quantities needed to compare IRFs are not simple scalars; therefore, the vector quantities needed to be adapted to simple statistical metrics.

The estimates of mean square and RMS of a random variable can be obtained with the following formulas,

$$\overline{\delta^2} = \frac{1}{n}\cdot\sum_{i=1}^{n} \delta_i^2 \qquad (6)$$

and

$$\hat{\sigma}_\delta = \sqrt{\overline{\delta^2}} \qquad (7)$$

respectively, where the δ_i, $i=1,\ldots,n$, are data from the random source. These formulas are unbiased and consistent (see Ang & Tang, 1975 [1] or Benjamin & Cornell, 1970 [3]).

To accommodate these simple statistical estimation formulas using IRF data, a measure of "distance" between neighboring system IRF vectors was defined based on a normalized inner product (see Moon & Wynn, pp. 101-103 [4] for an inner product space description of the discussion to follow). If $\mathbf{v}_1$ and $\mathbf{v}_2$ are a pair of real-valued n x 1 vectors, the normalized inner product (P) between $\mathbf{v}_1$ and $\mathbf{v}_2$ is

$$P = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\| \cdot \|\mathbf{v}_2\|} \tag{8}$$

where $\|\mathbf{v}\|$ denotes the norm of the vector **V**. Specifically

$$\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v} = \sum_{i=1}^{n} v_i^2 \tag{9}$$

where the $(i)^{th}$ element in vector **v** is v_i. Because of the normalization, this inner product is always in the interval [-1,1]. The angle, ϕ, between the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is defined as the inverse cosine of the normalized inner product (P).

$$\phi = \cos^{-1}(P) \tag{10}$$

In the following computations, ϕ will always be taken as the positive-valued angle whose cosine is P. A normalized distance between the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ can now be defined as

$$\delta = 2 \cdot \sin\left(\frac{\phi}{2}\right) \tag{11}$$

Because of the angle definitions, this quantity will always be nonnegative.

This framework for comparison of vectors permits the characterization of variation among a collection of random vectors. From the random vector **V,** v_i, i=1,…,M, is a sequence of real-valued n x 1 vectors. (**V** is an n x 1 vector of random variables). The sample mean of this collection is defined as

$$\bar{\mathbf{v}} = \frac{1}{M} \cdot \sum_{i=1}^{M} \mathbf{v}_i \tag{12}$$

The statistical variation of the collection of vectors $\mathbf{v}_i$, i=1,…,M, from the mean is now described. The distance of the normalized $(i)^{th}$ vector from the normalized mean is

$$\delta_i = 2 \cdot \sin\left(\frac{\phi_i}{2}\right) \qquad i=1,\dots,M \tag{13}$$

where

$$\phi_i = \cos^{-1}\left(\frac{\mathbf{v}_i \cdot \bar{\mathbf{v}}}{\|\mathbf{v}_i\| \cdot \|\bar{\mathbf{v}}\|}\right) \qquad i=1,\dots,M \tag{14}$$

The δ_i, i=1,…,M, are realizations of a random variable Δ. This quantity is the distance of a vector **v** to the sample mean. All the δ_i's are nonnegative, though the vectors occupy arbitrary locations in n-space. Therefore, a metric of the spread of vector locations in terms of the RMS is defined as

$$\hat{\sigma}_\Delta = \left[\frac{1}{M} \cdot \sum_{i=1}^{M} \delta_i^2\right]^{\frac{1}{2}} \tag{15}$$

This is an estimate of the RMS distance of vectors in **V** from their mean vector.

Example: if **V** is a 3 x 1 vector random source defined as

$$\mathbf{V} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} + 0.15 \cdot \begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{Bmatrix} \tag{16}$$

where the Z_i, i=1,2,3, are uncorrelated standard normal random variables, a sequence of random vectors $\mathbf{v}_i$, i=1,…,10, from the random source could be generated. The sample mean can also be calculated. The vector realizations and sample mean are shown in Figure 1.

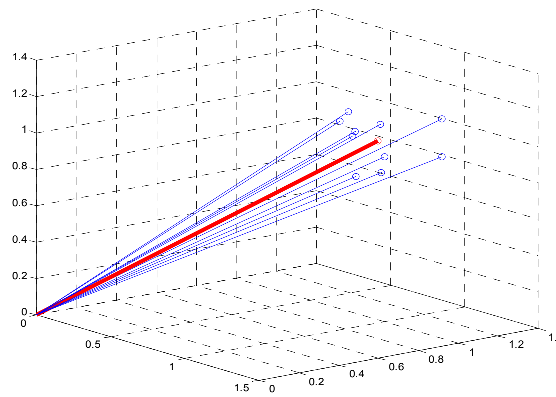

Figure 1: Ten realizations of a vector random source and their mean, where the mean is represented in red.

Figure 2 shows the normalized form of one of the vector realizations and the normalized sample mean (shown in Figure 1) in the plane created by the two vectors. The angle ϕ between the two vectors is shown, along with the distance between them.

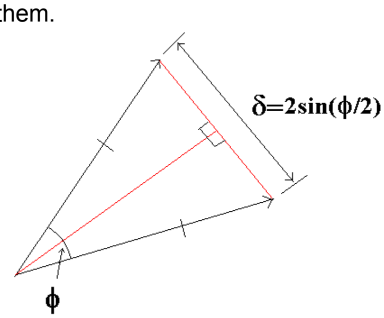

Figure 2: Distance between a realization vector and the sample mean vector

Figure 3 shows the end points of the 10 random vectors projected onto a plane perpendicular to the end of the mean vector. The distances δ_i, i=1,…,10, are approximately the distances shown in Figure 3.

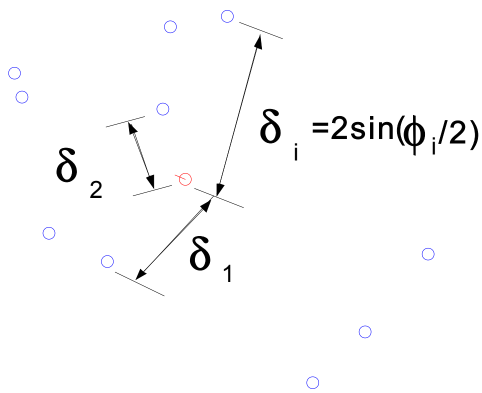

Figure 3: Distance from individual vectors to the sample mean vector. **End of example**

The framework of Eqs. (8) through (15) are now used to assess the variability of IRFs. Specifically, **h** is defined as an n x 1 vector random source of IRF realizations denoted as $\mathbf{h}_i$, i=1,…,M. The n-dimensional vectors, $\mathbf{h}_i$, i=1,…,M, are comparable to the three-dimensional vectors $\mathbf{v}_i$, i=1,…,M, defined above. The RMS deviation of the vectors in **h** from their mean is estimated using the framework defined in the example. This quantity is a metric of system variability.

The example computations shown are in three dimensions for visualization purposes, however, measured IRFs are n-dimensional. Nevertheless, the definitions established in this section are still applicable.

4. Test Setup

The objective of this investigation was to assess sources of variation in structural systems. Experiments on a real structure were chosen to accomplish this goal. For the sake of simplicity, the simple structure, shown in Figure 4, was chosen. The structure consists of a four-member aluminum frame bolted together using four steel ninety-degree angle brackets. The top member is 0.00635-m thick, 0.0508-m wide, and 0.5588-m long. The vertical members are 0.00635-m thick, 0.0508-m wide, and 0.3048-m long. The bottom member is 0.0127-m thick, 0.1524-m wide, and 0.6096-m long; the intent was that the bottom member simulates a rigid base. The steel angle brackets are 0.00635-m thick and 0.0508-m wide; the legs of the angles are 0.0635-m long. The entire structure was then centered on a piece of foam, 0.0254-m thick, 0.17145-m wide, and 0.2794-m long. The foam acted to create an approximately free-free boundary condition and to isolate the structure from the table on which it rested. By using this setup, the effects of the table on the characteristics measured in the test, were minimized. Each of the six larger bolts joining the three upper members were tightened to 54 N-m. The remaining connections were made with eight smaller screws. (These eight screws were not removed during any of the tests.)

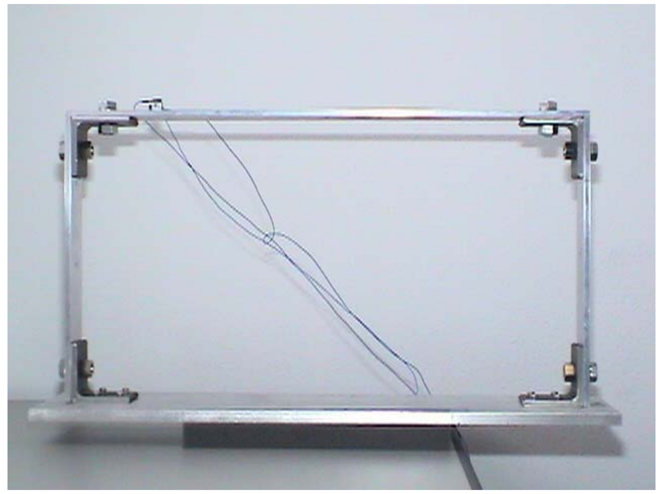

Figure 4: Sample test structure

Because the objective of one part of this investigation was to assess variations among nominally identical structures, six nominally identical top members and five nominally identical vertical members were fabricated. In the various experiments, random structures were created by selecting top and vertical members at random and using them to fabricate a structure like the one shown in Figure 4.

The data acquisition system used to collect the dynamic motion data was a four-channel Dactron Photon system, which was configured to measure four signals simultaneously at a sampling rate of 1280 samples per second. The system was excited using a PCB medium sized impact hammer with a rubber cap on a nylon tip. This means of excitation gave approximately five-decibels of roll-off for the 0 to 500 Hz range of interest with a peak impact level of roughly 155 N. All impacts were applied at the upper right corner of the structure. Three PCB 352C22 teardrop-style accelerometers were mounted on a small aluminum block, which was affixed to the upper left corner of the structure to measure responses in the three principle axes. The excitation and response locations were established using a modal analysis of the structure to find the locations which yielded responses with a high signal to noise ratio.

4.1 Experimental Procedure

This study considered five experiments each of which examines a different cumulative level of structural variation. In each experiment, five replicate tests were performed, with each test consisting of ten runs. A run was a set of five impacts and response measurements that were averaged to obtain a frequency response function (FRF). Each FRF was transformed into an IRF. The ten runs included in each test were used to estimate the RMS variation of IRFs from their sample mean using the approach and metric described in the previous section. Each of the five replicate tests sought to estimate the RMS value of a particular random source.

4.2 Summary of Experiments

The goal of each experiment was to assess variation in estimated IRFs caused by an accumulation of effects. The experiments are described below along with their associated random variation sources. Each test in experiments one through four was performed on a single structure assembled from random elements. The tests in experiment five were performed on different random structures for each run.

1. Rapid Succession Experiment – The runs of each test in this experiment were taken without pause in order to minimize the effects due to environmental changes. (Precautions were taken to insure the system was as consistent as possible. For example, the accelerometers were not moved during this experiment.) (Duration ~ 10 minutes)

Sources of variability are:
- Accelerometer noise
- Changes in structural joints and elements during response
- Level and location of excitation

2. Accelerometer Removal/Replacement Experiment – The accelerometers were removed and placed in nominally the same location and orientation between each run of this experiment. (Duration ~ 10 minutes)

Sources of variability are:
- Accelerometer placement
- All variation associated with experiment 1

3. Extended Observation Experiment – Each test in this experiment was run over several hours. (Duration ~ 2.5 hours)

Sources of variability are:
- Environmental changes (temperature, humidity) over duration of test
- All variation associated with experiment 1

4. Assembly/Disassembly Experiment – Each test in this experiment used the same structural elements in the same orientation but after each run the structure was disassembled and reassembled in the same orientation. (Duration ~ 1.5 hours)

Sources of variability are:
- Changes in boundary conditions, i.e. resetting of mating joint surfaces and joint configuration
- All variation associated with experiments 1, 2, and 3

5. Random Structure Experiment – Each run used a different random structure determined using a MATLAB code. The assembly and disassembly procedures were the same as the assembly/disassembly experiment.

Sources of variability are:
- Actual random changes in structural geometry
- Changes in material properties
- Changes in joint configuration (greater than experiment 4)
- All variation associated with experiments 1, 2, 3, and 4

5. Data Analysis

All data analyses described below were performed using MATLAB, (Version 6.1.0.450, Release 12.1). First, equations outlined in the Structural Dynamics section were used to estimate the system IRFs for each run, (where a run is an average of five impacts). The system IRFs were estimated by applying the inverse DFT to the system FRFs. In addition, the estimated IRFs were further modified through filtering. Specifically, the IRFs were bandpass filtered over the interval (35,75) Hz using a fourth order Butterworth filter in order to isolate structural motion in the first flexural mode of response. Figure 5 shows a typical structural FRF, while Figure 6 shows the corresponding filtered IRF.

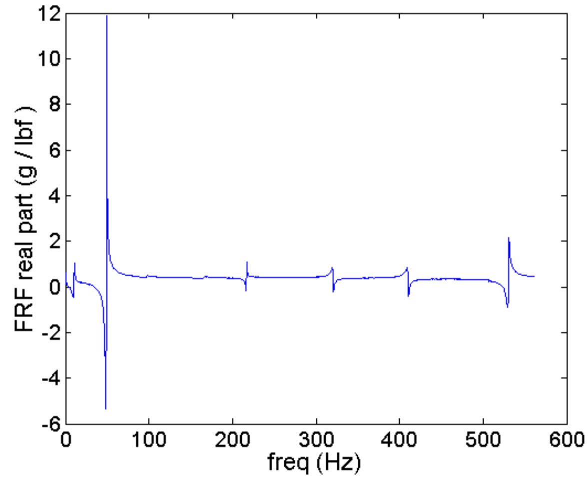

Figure 5: Unfiltered FRF of a sample random structure

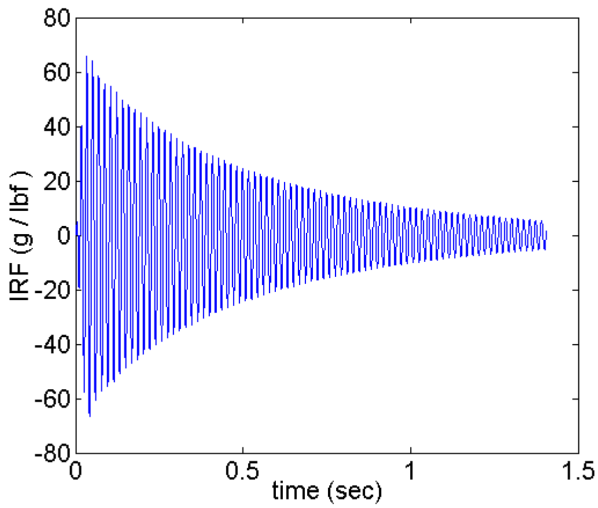

Figure 6: Filtered IRF of a sample random structure

Ten runs were performed for each test. The sample mean was calculated for each test as well as the RMS variation in IRFs in the manner established in section 3. Statistical Analysis. Five tests were completed for each experiment.

6. Results

A sequence of five experiments, described in the test setup section, was performed. Each experiment consisted of five tests. Each result presented below is an estimate of RMS structural variation. We anticipated that structural variation should be minimal in the Rapid Succession experiment and maximal in the Random Structures experiment. The results are presented in Figure 7, which is a logarithmic plot of RMS estimates associated with the five experiments. The results are ordered along the abscissa of the plot according to the indexing scheme established in the section 4.2 Summary of Experiments.

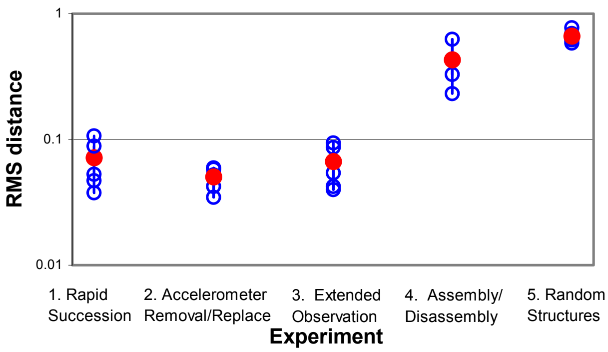

Figure 7: RMS values for variation corresponding to each experiment where an estimated RMS value for variation corresponding to a single test is shown by an open circle. A solid circle shows the sample RMS of the collection of individual RMS estimates. A solid line shows the span from the lowest to the greatest RMS estimate.

The variation was anticipated to increase from experiment 1 through 5 due to the cumulative effect of the variation introduced to the system. Whether environmental variation would have a larger effect on the system than accelerometer location and orientation was unclear; therefore, the relative magnitude of variations experienced during the Accelerometer Removal/Replacement and Extended Observation experiments was difficult to predict.

The data presented, in Figure 7, show that each of the first three experiments exhibit low variation associated with the conditions listed in the test summary. The Rapid Succession experiment exhibits a greater amount of variation than the other two experiments performed without rebuilding the structures. This trend in the data could be explained by a relatively small number of tests, too small to effectively measure the variances associated with each experiment. An important result to note is that the fourth and fifth experiments, in which the structure was disassembled and reassembled, produce an increase, of almost an order of magnitude, in estimated RMS variation. This result shows the high degree of variation associated with changing boundary conditions and structural components.

7. Conclusions

A simple frame structure was studied to develop a means of estimating variability in structural response. In this study, we found that structural characteristics and their estimates varied randomly due to various effects. The measurement system introduced variability in the form of transducer noise, level and location of excitation, and accelerometer placement. Further, environmental variation (e.g., temperature and humidity changes), introduce variability to the system. This study also found that disassembly and reassembly of a structure can introduce changes in constraints and boundary conditions, which correspond to variation in the system response. This variation is especially prevalent in structures with bolted joints. Lastly, this study found that nominally identical structures vary in geometry and material properties, resulting in variation of response from one randomly assembled structure to another.

We were able to establish a measure of random variation in structures applicable to any structure. By performing experiments on a simple structure, we were able to verify the applicability of this measure. These experiments allowed us to evaluate the effects of various sources of variation and gauge the degree of structural response variation associated with the accumulation of these sources of uncertainty.

Motivation for further work in this area can be found in many industries. In the age of mass production of complex structures, the ability to quantify the variation in structural response of a collection of nominally identical structures is important. This study can be used to quantify the variation in structural response. Further work could be done in analysis of the structure using finite elements to identify the parameter(s) that correspond to the variation in structural response. Incorporating varying parameters in an analysis can prove to be valuable in predicting a range of structural response for a collection of structures.

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